Discrete Mathematics
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Lecture -23
Partial Ordering

Hello everyone, welcome to this lecture on partial orderings. So, we will introduce the definition of partial ordering in this lecture. We will discuss Hasse diagram and we will compute with Topological sorting.

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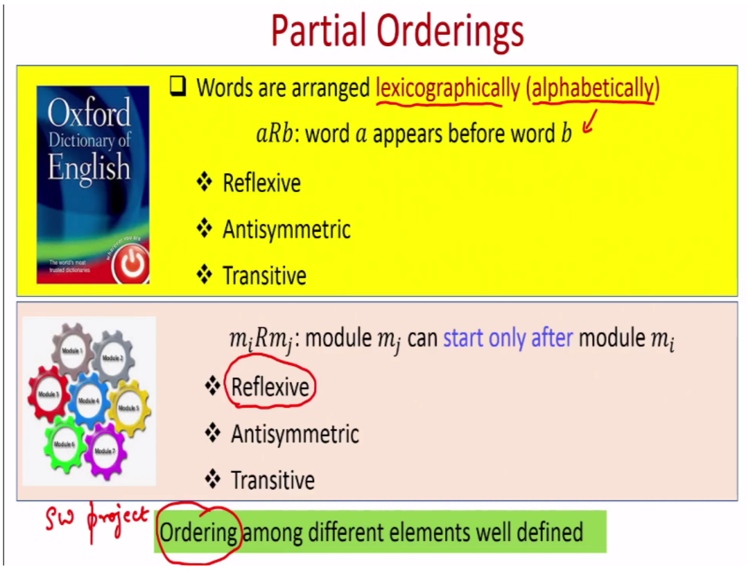

So, what is the partial ordering? So, if you consider a dictionary then the words in a dictionary are arranged alphabetically or we also say that the words are arranged lexicographically. And there is a very nice relationship which you can state that holds between the words or relationship holds with respect to the way in which the words are arranged in your dictionary. So, the relationship here is, I say that a word a in the dictionary is related to the word b in my dictionary provided the word a appears before the word b.

So, this alphabetical arrangement of the words can be considered as a relationship. And it turns out that this alphabetical arrangement of the words in the dictionary satisfies three properties. It satisfies the Reflexive property, it satisfies the Antisymmetric property and it satisfies the Transitive property.

Reflexive property because implicitly I can always say or I can always assume that a word always appear before itself. That is not true in the sense of the dictionary, but I can always have this implicit order. The alphabetical arrangement of the words satisfies antisymmetric properties because you cannot have two different words such that the word a appears before the word b and simultaneously the word b appears before the word a.

And this alphabetical arrangement of the words satisfies the transitive property because if you have the word b, appearing after the word a, and if you have the word c appearing after the word b, then you can say that the word c is appearing after the word a. So, that sense it is a transitive relation. It turns out that you can have several such relations which satisfy the property of being reflexive, antisymmetric and transitive.

So, for instance imagine, I have a big software project. And typically in a big software project you identify various modules, various components which are independent of each other and each of them can be executed by separate procedure. So, now imagine that I have several such modules and I have defined a relationship or a dependency between the modules by a relation R and I say that module m_i is related to the module m_j if there is a dependency on the module j for the module i. So, I define a relationship R where module i is related to module j provided module j can start only after module i is over. That means until and unless you are done with the module i, you cannot start the module j. That is a dependency relationship. Now again I can say here that this dependency relationship, is reflexive, it is anti symmetric and it is transitive.

It is reflexive in the sense I can always simply assume that a module always depends on itself. It is an implicit dependency. This dependency relationship is anti symmetric because I cannot have two separate modules which are dependent on each other. If that is the case if that situation happens in your software project then it will lead to a state of a dead lock. So, for example, module 1 depends on module 2 and module 2 depends on module 1 you cannot start both of the any of them. So, that is why this relationship will be anti symmetric.

And is relationship is transitive. If module 2 depends on module 1 and if module 3 depends on module 2, implicitly it means that module 3 depends on module 1 as well. So, I have given you examples of two relations each of them satisfies the reflexive, anti symmetric and transitive properties and the essence of both this examples is the following.

You have a well defined ordering among different elements. If I take the first example my elements were the words of the dictionary and there is a way to ordering, the alphabetical order. If I take the second example, my elements of the set were the modules of the software project and there is a well defined ordering.

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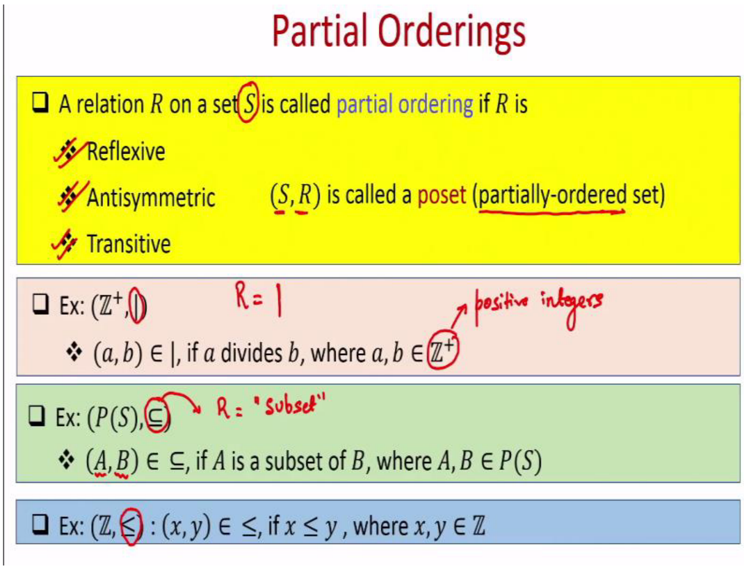

So, let us now generalize this theory. So, we now have we are now going to define a special type of relation which we call as a partial ordering. So, you are given a set S over which a relation R is defined and it will be called as a partial ordering, if the relation is reflexive, antisymmetric and transitive. In that case, the set S along with the relation R is called a poset. The full form of poset is partially ordered set.

Let me give you some more examples of partial ordering here. So, I consider the set of all positive integers, so this set $\mathbb{Z}^+$ is the set of positive integers. And I defined a relation divides which I am denoting by $|$. So, my relation R is the $divides$ ($|$) relationship.

And I say that $(a, b) \in | \ if \ a|b$ and $a, b \in \mathbb{Z}^+$. Otherwise, a is not related to b. So, now it is easy to see that this relationship is reflexive because every positive integer divides itself. This relationship is antisymmetric, because you cannot have two different positive integers simultaneously dividing each other if a divides b as well as b divides a then that is possible only when both the integers are same.

And if $| \ if \ a|b$ and $b|c$, then you have $a|c$. So, it satisfies the transitivity property. So, this is an example of a partial ordering. Now, let me define another relation here my relation here is $\subseteq$. My R here is the subset relationship, which I am denoting by $\subseteq$. And my elements are the elements of the power set of a set. So, the relation is not defined over the set S. I stress that the relation is defined over the power set of S.

So, my elements are here subsets of S and I say that a subset A is related to the subset B, if$A \subseteq B$. That is my relation. Again this relation satisfies the reflexive property because $A \subseteq A$. It satisfies the antisymmetric property because you cannot have two different subsets A and B, $A \subseteq B$ and $B \subseteq A$, because that is the case that means $A = B$. And it satisfies the requirement of a transitive relation. If $A \subseteq B$ and $B \subseteq C$, then that means that $A \subseteq C$. So, this is an example of partial ordering.

Similarly, if I take the set of integers, $\mathbb{Z}$, and if my relationship is the less than equal to relationship where integer x is related to integer y provided $x \leq y$. Then again this satisfies the reflexive property, antisymmetric property and transitive property and hence this is an example of a partial ordering.

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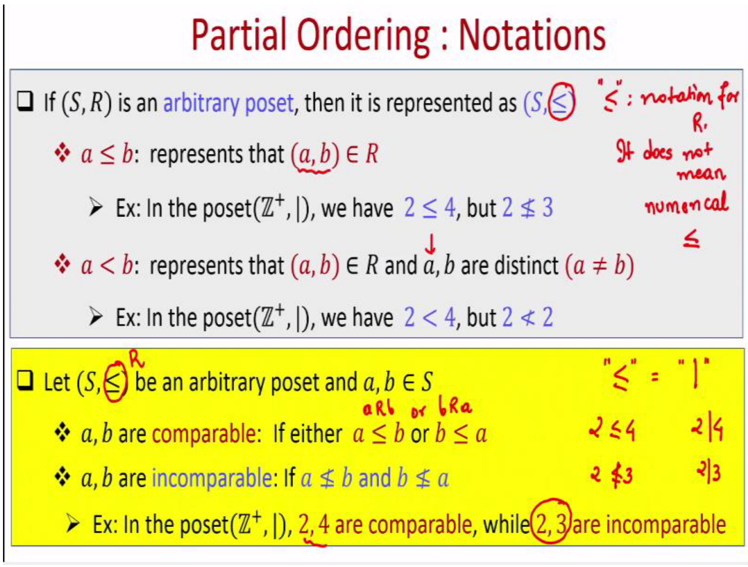

So, now if you are given an arbitrary poset instead of using the notation R for the relation, I use the abstract notation $\leq$. I Stress here that is $\leq$is just a notation. It is just a substitute for R, it is notation for R. It does not mean numerical less than equal to, that is very important. That means when I am writing $a \leq b$, that does not mean that a is numerically less than equal to b. That just mean, that the element a is related to element b as per my relation R.

So, for example, if I take the partial ordering where my relation was the divide (|) relationship then $2 \leq 4$. Again, do not get confused by the numerical interpretation. Again numerically indeed 2 is less than equal to 4 but less than equal 2 here stands for the divide relationship namely 2 |4, but 2 is not less than equal to 3. Because, we are not numerically following the interpretation here, 2 does not divide 3. That is why 2 is not less than equal to 3.

Now we also use this abstract notation $<$ and again, this is not a numerical representation. It is just use to represent the fact that a is related to b but $a \neq b$. So, that is a case I use the notation a less than b. So, for instance, if I take the | relationship we have 2 less than 4, because indeed 2 divides 4, and 2 is not equal to 4, whereas we have 2 not less than 2, even though 2 is less than equal to 2, because 2 divides 2. But since 2 and 2 are same I cannot say that 2 is less than 2. So, that is the abstract notation that we are now going to follow for the rest of our discussion on partial ordering. Less than equal to is not numerical less than equal to, less than is not the numerical less than.

So, imagine you are given an arbitrary poset that so this less than equal to is an arbitrary relation R, which is a reflexive, antisymmetric and transitive. Now you take any two elements from the set S. They will be called comparable if $a \leq b$ or $b \leq a$, incomparable otherwise. So, again to demonstrate these two concepts let us consider the divide relationship, that means you are less than equal to relationship is the divides relationship.

Then you have $2 \leq 4$, because $2|4$. So, 2 and 4 are comparable. Comparable in the sense that they have there is definitely a relationship between 2 and 4. Either 2 is related to 4 or 4 is related to 2. But 2 is not related to 3, because 2 does not divide 3, that is why we will say that 2 and 3 are in comparable. So, in a partial ordering it is not necessary that you take any pair of elements and they are comparable. You may not have any relationship among them as per the relation R that you are considered.

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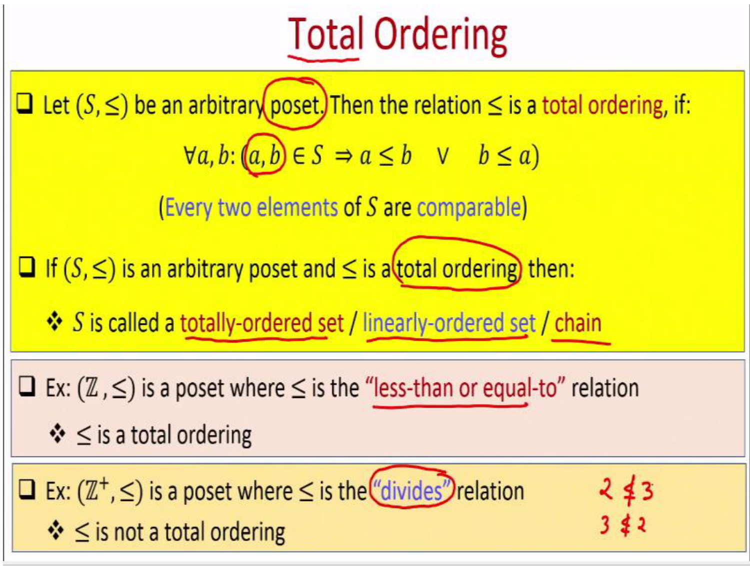

So, that brings us to the definition of what we call as a total ordering. And a total ordering is a special type of poset or partial ordering where you take any pair of elements from your set S, they will be comparable. That means either a will be related to b or b is related to a and that is why the name total ordering because you do not have any pair of incomparable elements. Whereas partial ordering the name partial denotes there, that you have ordering which is only partial. That means you may have a pair of incomparable elements and your relation R whereas a total ordering means you do not have any pair of incomparable elements. $\forall a, b \in S, a \leq b$ or $b \leq a$. So, that is why when your poset is a totally ordering, that means your relation is a total ordering then the set S is called as a totally ordered set. It is not called a partially ordered set.

In partial order set you might have the possibility of existence of incomparable elements. But in a totally ordered set you have relationship present between every pair of elements in the set. A total order set is also called as a linearly ordered set or a chain. Why it is called a chain or a linearly ordered set will be clear soon. So, let us see some examples of total ordering.

If I consider the less than equal to relationship namely, the numerical less than equal to relationship over the set of integers, then it is a total ordering. You take any pair of integers numerically either the first integer will be less than equal to the second integer or the second integer will be less than equal to the first integer. Whereas if you take the divides relationship, where a is related to b, provided a divides b then this is not a total ordering. Because 2 is not less than equal to 3 and 3 or also is not less than equal to 2. So, 2 and 3 are incomparable elements.

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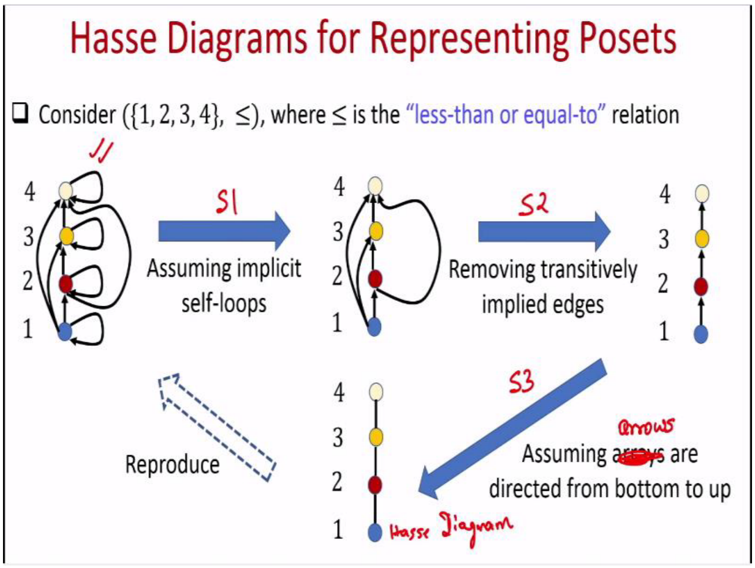

So, it turns out that we can represent partial ordering or posets by a very specific type of diagrams which are called as Hasse diagrams. And this is possible provided, this makes this is more interesting for posets where the relation is to find over a finite set. So, what exactly is this Hasse diagram? So, let me demonstrate this Hasse diagram with this less than equal to relationship which is the numerical less than equal to relationship defined over the set $S = \{1, 2, 3, 4\}$.

So, this will be the directed graph for your relationship less than equal to. Since 1 is related to 1, I have the self loop at the node 1, 2 is related to 2, so I have the self loop at the node 2. Similarly, I have the self loop at 3 because 3 is related to 3 and I have the self loop at 4, because 4 is related to 4. I have a directed edge from 1 to 2, because 1, 2 is present in the relationship. I have a directed edge from 1 to 3 because 1 is related to 3 and so on.

So, all the directed edge which are supposed to be present in the relation are there in this graph. Now what I can say here is that there is no point of explicitly writing down or stating the self loops. Because I can say that since my relationship is reflexive anyhow, I can always say that the self loops are implicitly present in my diagram. No need to unnecessarily represent them and make the diagram untidy.

So, if I remove the self loops and assume that my, self loops are always implicitly present, then my diagram looks little bit better. Next what I can do is I can remove the transitively implied edges from this diagram and say that hey, since my relation is anyhow transitive, I can remove the edge present from the node 1 to 3. Because I can say that since 1 to 2 is present and 2 to 3 is present anyhow 1 to 3 will be present in my diagram. So, why to again explicitly represented in the diagram. So, I can remove all the transitively implied edges and my diagram simplifies further.

So, what I am doing is in each stage, I am trying to make my diagram more and more cleaner, tidy and try to remove all unnecessary information or redundant information, which I am not supposed to explicitly state in my graph of the relation of a partial ordering.

Now what I can say is that I can say that I make the assumption here that the arrays here, so sorry it is not arrays, this is arrows. So, I can make the assumption here that the arrows are always directed from bottom to up and that will take care of the direction of the edges as well and my graph becomes further simplified to this diagram. And now there is no more of information, which I can remove from this graph and say that it still represents my original relationship that means what I mean by that is if I take this graph which I obtained by step 1 and then followed by step 2 and followed by step 3 then if you give me just this graph I can reproduce the original graph. How can I reproduce the original graph?

As per my definition, I will say that arrows are always pointed upwards. Then as per my assumption the self-loops are always there. But and as per my assumption all the transitively implied edges are also there in my graph. That means if you give me the graph, this final graph which will be called as the Hasse Diagram here. If you give me the Hasse diagram here, I can reproduce the entire original graph for the Partial ordering that you were given here right. So, that is how you construct a Hasse diagram for partial order.

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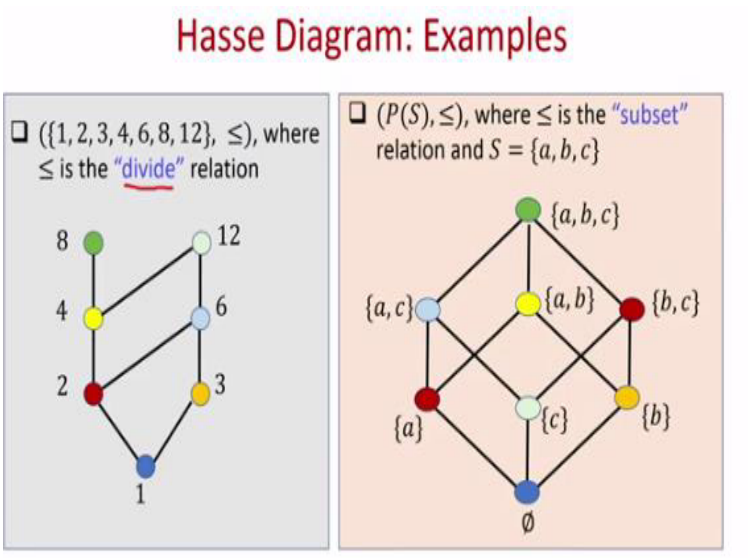

So, let us see another example here. So, you are given the divide (|) relationship. So, your $\leq$ is the divider relationship. So, again, we can start with our directed graph with the nodes 1, 2, 3, 4, 6, 8, 12, I can have all the self loops. I will have the transitively implied edges and so on and then if I remove all the self loops all the transitively implied edges, and if I remove the direction of the edges assuming that the arrows are always pointed from bottom to up, then this will be the Hasse diagram that I will obtain. This will be the minimum piece of information, which I need to retain in my graph to recover back the original diagram of $\leq$ or the $|$ relationship over this set $S = \{1, 2, 3, 4, 6, 8, 12\}$.

Let us see another example where the less than equal to relationship is the subset relation. And relation is defined over the power set of $S = \{a, b, c\}$ not over the set {a, b, c}, remember. So, how many elements will be there in the power set of {a, b, c}? So, since set S has 3 elements the cardinality of its power set will be 2^3. There will be 8 subsets, $\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$. So, these are the 8 subsets. Again, I have removed all the self loops. I have removed all the transitively implied edges and I have removed the direction of the edges.

So, for instance, I have not added the edge from the subset ϕ to the subset $\{a, c\}$, because that is transitively implied. Because ϕ is anyhow a subset of a which is represented by this undirected edge and undirected edge always have an implicit direction associated with it. And a is a subset of the subset $\{a, c\}$. Again, the direction is not explicitly mentioned here, but as per my assumption the directions are always upward. And as per my assumption the transitively edges are not explicitly stated in the graph. That means I have an implicit edge from ϕ to the subset $\{a, c\}$. Because indeed the subset ϕ is a subset of the subset $\{a, c\}$. But I do not need to explicitly add it in the graph. I can remove it. So, this is the minimum piece of information which I need to have in my graph to recover back the entire directed graph of the subset relationship over the power set of $\{a, b, c\}$. So, this will be the Hasse diagram of the subset relationship over the set $\{a, b, c\}$. So, why I am drawing all this Hasse diagram and all? Because that helps us to understand the next few concepts, which we are going to describe next.

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